

Stable P-symmetric closed characteristics on partially symmetric compact convex hypersurfaces

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Dedicate to Professor Rou-Huai Wang's 90th birth anniversary

Abstract

In this paper, let $n \geq 2$ be an integer, $P = \text{diag}(-I_{n-\kappa}, I_\kappa, -I_{n-\kappa}, I_\kappa)$ for some integer $\kappa \in [0, n-1)$, and $\Sigma \subset \mathbf{R}^{2n}$ be a partially symmetric compact convex hypersurface, i.e., $x \in \Sigma$ implies $Px \in \Sigma$. We prove that if Σ is (r, R) -pinched with $\frac{R}{r} < \sqrt{\frac{5}{3}}$, then Σ carries at least two geometrically distinct P-symmetric closed characteristics which possess at least $2n - 4\kappa$ Floquet multipliers on the unit circle of the complex plane.

Key words: Compact convex hypersurfaces, stable P-symmetric closed characteristics, P-index iteration, Hamiltonian system.

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1 Introduction and main results

In this paper, we consider the stability of P-symmetric closed characteristics on partially symmetric hypersurfaces in \mathbf{R}^{2n} . Let Σ be a C^3 compact hypersurface in \mathbf{R}^{2n} , bounding a strictly convex

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compact set U with non-empty interior, where $n \geq 2$. We denote the set of all such hypersurfaces by $\mathcal{H}(2n)$. Without loss of generality, we suppose U contains the origin. We consider closed characteristics (τ, y) on Σ , which are solutions of the following problem

$$\begin{cases} \dot{y}(t) = JN_\Sigma(y(t)), & y(t) \in \Sigma, \forall t \in \mathbf{R}, \\ y(\tau) = y(0), \end{cases} \quad (1.1)$$

where $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$, I_n is the identity matrix in \mathbf{R}^n and $N_\Sigma(y)$ is the outward normal unit vector of Σ at y normalized by the condition $N_\Sigma(y) \cdot y = 1$. Here $a \cdot b$ denotes the standard inner product of $a, b \in \mathbf{R}^{2n}$. A closed characteristic (τ, y) is *prime* if τ is the minimal period of y . Two closed characteristics (τ, x) and (σ, y) are *geometrically distinct*, if $x(\mathbf{R}) \neq y(\mathbf{R})$. We denote by $\mathcal{J}(\Sigma)$ the set of all closed characteristics (τ, y) on Σ with τ being the minimal period of y . For any $s_i, t_i \in \mathbf{R}^{k_i}$ with $i = 1, 2$, we denote by $(s_1, t_1) \diamond (s_2, t_2) = (s_1, s_2, t_1, t_2)$. Fixing an integer κ with $0 \leq \kappa < n - 1$, let $P = \text{diag}(-I_{n-\kappa}, I_\kappa, -I_{n-\kappa}, I_\kappa)$ and $\mathcal{H}_\kappa(2n) = \{\Sigma \in \mathcal{H}(2n) \mid x \in \Sigma \text{ implies } Px \in \Sigma\}$. For $\Sigma \in \mathcal{H}_\kappa(2n)$, let $\Sigma(\kappa) = \{z \in \mathbf{R}^{2\kappa} \mid 0 \diamond z \in \Sigma\}$, where 0 is the origin in $\mathbf{R}^{2n-2\kappa}$. As in [DoL1], A closed characteristic (τ, y) on $\Sigma \in \mathcal{H}_\kappa(2n)$ is *P-asymmetric* if $y(\mathbf{R}) \cap Py(\mathbf{R}) = \emptyset$, it is *P-symmetric* if $y(\mathbf{R}) = Py(\mathbf{R})$ with $y = y_1 \diamond y_2$ and $y_1 \neq 0$, or it is *P-fixed* if $y(\mathbf{R}) = Py(\mathbf{R})$ and $y = 0 \diamond y_2$, where $y_1 \in \mathbf{R}^{2(n-\kappa)}$, $y_2 \in \mathbf{R}^{2\kappa}$. We call a closed characteristic (τ, y) is *P-invariant* if $y(\mathbf{R}) = Py(\mathbf{R})$. Then a P-invariant closed characteristic is P-symmetric or P-fixed.

Let $j : \mathbf{R}^{2n} \rightarrow \mathbf{R}$ be the gauge function of Σ , i.e., $j(\lambda x) = \lambda$ for $x \in \Sigma$ and $\lambda \geq 0$, then $j \in C^3(\mathbf{R}^{2n} \setminus \{0\}, \mathbf{R}) \cap C^0(\mathbf{R}^{2n}, \mathbf{R})$ and $\Sigma = j^{-1}(1)$. Fix a constant $\alpha \in (1, +\infty)$ and define the Hamiltonian $H_\alpha : \mathbf{R}^{2n} \rightarrow [0, +\infty)$ by

$$H_\alpha(x) := j(x)^\alpha$$

Then $H_\alpha \in C^3(\mathbf{R}^{2n} \setminus \{0\}, \mathbf{R}) \cap C^0(\mathbf{R}^{2n}, \mathbf{R})$ is convex and $\Sigma = H_\alpha^{-1}(1)$. It is well known that the problem (1.1) is equivalent to the following given energy problem of the Hamiltonian system

$$\begin{cases} \dot{y}(t) = JH'_\alpha(y(t)), & H_\alpha(y(t)) = 1, \forall t \in \mathbf{R}, \\ y(\tau) = y(0). \end{cases} \quad (1.2)$$

Denote by $\mathcal{J}(\Sigma, \alpha)$ the set of all solutions (τ, y) of the problem (1.2), where τ is the minimal period of y . Note that elements in $\mathcal{J}(\Sigma)$ and $\mathcal{J}(\Sigma, \alpha)$ are in one to one correspondence with each other. Let $(\tau, y) \in \mathcal{J}(\Sigma, \alpha)$. We call the fundamental solution $\gamma_y : [0, \tau] \rightarrow Sp(2n)$ with $\gamma_y(0) = I_{2n}$ of the linearized Hamiltonian system

$$\dot{z}(t) = JH''_\alpha(y(t))z(t), \quad \forall t \in \mathbf{R}.$$

the *associated symplectic path* of (τ, y) . The eigenvalue of $\gamma_y(\tau)$ are called *Floquet multipliers* of (τ, y) . By Proposition 1.6.13 of [Eke1], the Floquet multipliers with their multiplicities and Krein type numbers of $(\tau, y) \in \mathcal{J}(\Sigma, \alpha)$ do not depend on the particular choice of the Hamiltonian function in (1.2). As in Chapter 15 of [Lon1], for any symplectic matrix M , we define the elliptic height $e(M)$ of M by the total algebraic multiplicity of all eigenvalues of M on the unit circle \mathbf{U} in the complex plane \mathbf{C} . And for any $(\tau, y) \in \mathcal{J}(\Sigma, \alpha)$ we define $e(\tau, y) = e(\gamma_y(\tau))$, and call (τ, y) *elliptic* or *hyperbolic* if $e(\tau, y) = 2n$ or $e(\tau, y) = 2$, respectively.

As in Definition 5.1.6 of [Eke1], a C^3 hypersurface Σ bounding a compact convex set U , containing 0 in its interior is (r, R) -*pinched*, with $0 < r \leq R$, if:

$$|y|^2 R^{-2} \leq \frac{1}{2}(H_2''(x)y, y) \leq |y|^2 r^{-2}, \quad \forall x \in \Sigma.$$

For the existence, multiplicity and stability of closed characteristics on convex compact hypersurfaces in \mathbf{R}^{2n} we refer to [Rab1, Wei1, EkL1, EkL2, Gir1, EkH1, Szu1, LLZ1, LoZ1, WHL1] and references therein. It is very interesting to consider closed characteristics on hypersurfaces with special symmetries. [Wan1, Liu2, Zha1] studied the multiplicity of closed characteristics with symmetries on convex compact hypersurfaces without pinching conditions. For the stability problem of closed characteristics with symmetries, in [HuS1] of 2009, Hu and Sun studied the index theory and stability of periodic solutions in Hamiltonian systems with symmetries. As application they studied the stability of figure-eight orbit due to Chenciner and Montgomery in the planar three-body problems with equal masses. In [Liu1], Liu studied the stability of symmetric closed characteristics on central symmetric compact convex hypersurfaces under a pinching condition. In [DoL2], Dong and Long proved that there exists at least one P -invariant closed characteristic which possesses at least $2n - 4\kappa$ Floquet multipliers on the unit circle of the complex plane. In this paper, we can obtain two such closed characteristics under a pinching condition:

Theorem 1.1. *Assume $\Sigma \in \mathcal{H}_\kappa(2n)$ and $0 < r \leq |x| \leq R$, $\forall x \in \Sigma$ with $\frac{R}{r} < \sqrt{\frac{5}{3}}$. Then there exist at least two geometrically distinct P -symmetric closed characteristics which possess at least $2n - 4\kappa$ Floquet multipliers on the unit circle of the complex plane.*

Remark 1.2. In the above Theorem 1.1, let $\kappa = 0$, the P -symmetric closed characteristic is just symmetric and the P -fixed closed characteristics vanish, so Theorem 1.1 covers Theorem 1.1 of [Liu1].

In this paper, let $\mathbf{N}, \mathbf{N}_0, \mathbf{Z}, \mathbf{Q}, \mathbf{R}$ and \mathbf{C} denote the sets of natural integers, non-negative integers, integers, rational numbers, real numbers and complex numbers respectively. Denote by $a \cdot b$ and $|a|$ the standard inner product and norm in \mathbf{R}^{2n} . Denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the standard L^2 -inner

product and L^2 -norm. For an S^1 -space X , we denote by X_{S^1} the homotopy quotient of X by S^1 , i.e., $X_{S^1} = S^\infty \times_{S^1} X$, where S^∞ is the unit sphere in an infinite dimensional *complex* Hilbert space. we define the functions

$$[a] = \max \{k \in \mathbf{Z} \mid k \leq a\}, \quad \{a\} = a - [a], \quad E(a) = \min \{k \in \mathbf{Z} \mid k \geq a\}, \quad \phi(a) = E(a) - [a].$$

Specially, $\phi(a) = 0$ if $a \in \mathbf{Z}$, and $\phi(a) = 1$ if $a \notin \mathbf{Z}$. We use \mathbf{Q} coefficients for all homological modules.

2 A variational structure for P-invariant closed characteristics

In the rest of this paper, we fix a $\Sigma \in \mathcal{H}_\kappa(2n)$. In this section, we review a variational structure for P-invariant closed characteristics established in [Liu2].

As in [Liu2], we associate with U a convex function H_a . Consider the fixed period problem

$$\begin{cases} \dot{x}(t) = JH'_a(x(t)), \\ x(1/2) = Px(0). \end{cases} \quad (2.1)$$

Then by Proposition 2.2 of [Liu2], nonzero solutions of (2.1) are in one to one correspondence with P-symmetric closed characteristics with period $\tau < a$ and P-fixed closed characteristics with period $\frac{\tau}{2} < \frac{a}{2}$. Let

$$\begin{aligned} L_\kappa^2\left(0, \frac{1}{2}\right) &= \{u = u_1 \diamond u_2 \in L^2((0, \frac{1}{2}), \mathbf{R}^{2n}) \mid u_1 \in L^2((0, \frac{1}{2}), \mathbf{R}^{2n-2\kappa}), \\ &\quad u_2 \in L^2((0, \frac{1}{2}), \mathbf{R}^{2\kappa}), u(\frac{1}{2}) = Pu(0), \int_0^{\frac{1}{2}} u_2(t)dt = 0\} \end{aligned} \quad (2.2)$$

Define a linear operator $\Pi_\kappa : L_\kappa^2(0, \frac{1}{2}) \rightarrow L_\kappa^2(0, \frac{1}{2})$ by

$$\begin{aligned} (\Pi_\kappa u)(t) &= x_1(t) \diamond x_2(t), \\ x_1(t) &= \int_0^t u_1(\tau)d\tau - \frac{1}{2} \int_0^{\frac{1}{2}} u_1(\tau)d\tau, \\ x_2(t) &= \int_0^t u_2(\tau)d\tau - 2 \int_0^{\frac{1}{2}} dt \int_0^t u_2(\tau)d\tau, \end{aligned}$$

for any $u = u_1 \diamond u_2 \in L_\kappa^2(0, \frac{1}{2})$.

The corresponding Clarke-Ekeland dual action functional is defined by

$$\Psi_a(u) = \int_0^{\frac{1}{2}} \left(\frac{1}{2} Ju \cdot \Pi_\kappa u + G_a(-Ju) \right) dt, \quad (2.3)$$

where G_a is the Fenchel transform of H_a defined by $G_a(x) = \sup \{x \cdot y - H_a(y) \mid y \in \mathbf{R}^{2n}\}$. By Proposition 2.6 of [Liu2], Ψ_a is $C^{1,1}$ on $L_\kappa^2(0, \frac{1}{2})$ and satisfies the Palais-Smale condition. Suppose

x is a solution of (2.1). Then $u = \dot{x}$ is a critical point of Ψ_a . Conversely, suppose u is a critical point of Ψ_a . Then there exists a unique $\xi \in \mathbf{R}^{2n}$ such that $\Pi_\kappa u - \xi$ is a solution of (2.1). In particular, solutions of (2.1) are in one to one correspondence with critical points of Ψ_a . Moreover, $\Psi_a(u) < 0$ for every critical point $u \neq 0$ of Ψ_a .

Suppose u is a nonzero critical point of Ψ_a . Then the formal Hessian of Ψ_a at u is defined by

$$Q_a(v, v) = \int_0^{\frac{1}{2}} (Jv \cdot \Pi_\kappa v + G_a''(-Ju)Jv \cdot Jv) dt, \quad (2.4)$$

which defines an orthogonal splitting $L_\kappa^2(0, \frac{1}{2}) = E_- \oplus E_0 \oplus E_+$ of $L_\kappa^2(0, \frac{1}{2})$ into negative, zero and positive subspaces. The index of u is defined by $i(u) = \dim E_-$ and the nullity of u is defined by $\nu(u) = \dim E_0$. cf. Definition 2.10 of [Liu2].

Note that we can identify $L_\kappa^2(0, \frac{1}{2})$ with the space $\{u \in L^2(\mathbf{R}/\mathbf{Z}, \mathbf{R}^{2n}) \mid u|_{(0, 1/2)} \in L_\kappa^2(0, \frac{1}{2}), u(t + \frac{1}{2}) = Pu(t)\}$. Then we have a natural S^1 -action on $L_\kappa^2(0, \frac{1}{2})$ defined by $\theta * u(t) = u(\theta + t)$, for all $\theta \in S^1 \equiv \mathbf{R}/\mathbf{Z}$ and $t \in \mathbf{R}$. By Lemma 2.8 of [Liu2], Ψ_a is S^1 -invariant. Hence if u is a critical point of Ψ_a , then the whole orbit $S^1 \cdot u$ is formed by critical points of Ψ_a . Denote by $\text{crit}(\Psi_a)$ the set of critical points of Ψ_a . Then $\text{crit}(\Psi_a)$ is compact by the Palais-Smale condition.

Recall that for a principal $U(1)$ -bundle $E \rightarrow B$, the Fadell-Rabinowitz index (cf. [FaR1]) of E is defined to be $\sup\{k \mid c_1(E)^{k-1} \neq 0\}$, where $c_1(E) \in H^2(B, \mathbf{Q})$ is the first rational Chern class. For a $U(1)$ -space, i.e., a topological space X with a $U(1)$ -action, the Fadell-Rabinowitz index is defined to be the index of the bundle $X \times S^\infty \rightarrow X \times_{U(1)} S^\infty$, where $S^\infty \rightarrow CP^\infty$ is the universal $U(1)$ -bundle.

As on Page 199 of [Ekel], we choose some $\alpha \in (1, 2)$ and associate with U a convex function $H(x) = j(x)^\alpha, \forall x \in \mathbf{R}^{2n}$. Consider the fixed period problem

$$\begin{cases} \dot{x}(t) = JH'(x(t)), \\ x(\frac{1}{2}) = Px(0). \end{cases}$$

The corresponding Clarke-Ekeland dual action functional on $L_\kappa^2(0, \frac{1}{2})$ is defined by

$$\Psi(u) = \int_0^{\frac{1}{2}} \left(\frac{1}{2} Ju \cdot \Pi_\kappa u + H^*(-Ju) \right) dt, \forall u \in L_\kappa^2\left(0, \frac{1}{2}\right),$$

where H^* is the Fenchel transform of H .

For any $\iota \in \mathbf{R}$, we denote by

$$\Psi^{\iota-} = \left\{ w \in L_\kappa^2(0, \frac{1}{2}) \mid \Psi(w) < \iota \right\}. \quad (2.5)$$

As in Section 2 of [Liu2], we define

$$c_i = \inf\{\delta \in \mathbf{R} \mid \hat{I}(\Psi^{\delta-}) \geq i\}. \quad (2.6)$$

where \hat{I} is the Fadell-Rabinowitz index defined above. Then

$$c_1 \leq c_2 \leq \cdots c_i \leq c_{i+1} \leq \cdots < 0.$$

By Propositions 2.15 and 2.16 of [Liu2], we have

Proposition 2.1. *Every c_i is a critical value of Ψ . If $c_i = c_j$ for some $i < j$, then there are infinitely many geometrically distinct P -invariant closed characteristics on Σ .*

Proposition 2.2. *For every $i \in \mathbf{N}$, there is a critical point u_α of Ψ found in Proposition 2.1 such that*

$$\Psi(u_\alpha) = c_i, \quad C_{S^1, 2i-2}(\Psi_a, S^1 \cdot u) \neq 0 \quad (2.7)$$

where u is a critical point of Ψ_a corresponding to u_α in the natural sense. In particular, we have $i(u) \leq 2(i-1) \leq i(u) + \nu(u) - 1$.

3 Index iteration theory for P-symmetric closed characteristics

In this section, we review the index iteration theory for P-symmetric closed characteristics which was studied in Section 3 of [Liu2].

Note that if (τ, y) is P-symmetric, then $((2m-1)\tau, y)$ is P-symmetric for any $m \in \mathbf{N}$. Thus $((2m-1)\tau, y)$ corresponds to a critical point of Ψ_a via Propositions 2.2 and 2.6 of [Liu2], we denote it by u^{2m-1} . Recall that the action of a closed characteristic (τ, y) is defined by (cf. P.190 of [Eke1])

$$A(\tau, y) = \frac{1}{2} \int_0^\tau (Jy \cdot \dot{y}) dt.$$

Lemma 3.1. (cf. Lemma 3.1 of [Liu2]) *Suppose u^{2m-1} is a nonzero critical point of Ψ_a such that u corresponds to P-symmetric closed characteristic (τ, y) . Let $H_2(x) = j^2(x)$, where j is the gauge function of Σ . And by (21) in P.191 of [Eke1], $\tau_2 = A(\tau, y)$. Then $i(u^{2m-1})$ equals the index of the following quadratic form*

$$q_{(2m-1)\tau_2/2, \kappa}(v, v) := \int_0^{(2m-1)\tau_2/2} [(Jv, \Pi_{(2m-1)\tau_2/2, \kappa} v) + (H_2''(y(t))^{-1} Jv, Jv)] dt. \quad (3.1)$$

where $v \in L_\kappa^2(0, (2m-1)\tau_2/2)$, the definitions of $q_{(2m-1)\tau_2/2, \kappa}$, $\Pi_{(2m-1)\tau_2/2, \kappa}$, $L_\kappa^2(0, (2m-1)\tau_2/2)$ are as in section 3 of [DoL2]. Moreover, we have $\nu(u^{2m-1}) = \nu(q_{(2m-1)\tau_2/2, \kappa}) - 1$.

Now we consider the linear Hamiltonian system

$$\begin{cases} \dot{\xi}(t) = JA(t)\xi, \\ A(t + \tau_2/2) = PA(t)P. \end{cases} \quad (3.2)$$

where $A(t) = H_2''(y(t))$. Denote by $i_P^E(A, k)$ and $\nu_P^E(A, k)$ the index and nullity of the k -th iteration of the system (3.2) defined by Dong and Long (cf. Definition 3.4 of [DoL2]). Denote by $i_{P,1}(\gamma_A^{k,P})$ and $\nu_{P,1}(\gamma_A^{k,P})$ the P-index and P-nullity of the k -th iteration of the system (3.2) defined by Dong and Long (cf. Section 3 of [DoL1]), where γ_A is the fundamental solution of (3.2) with $\gamma_A(0) = I_{2n}$. Then we have

Theorem 3.2. (cf. Theorem 3.2 of [Liu2]) *If u^{2m-1} is a nonzero critical point of Ψ_a such that u corresponds to P -symmetric closed characteristic (τ, y) . Then we have*

$$\begin{aligned} i(u^{2m-1}) &= i_P^E(A, 2m-1) = i_{P,1}(\gamma_A^{2m-1,P}) - \kappa, \\ \nu(u^{2m-1}) &= \nu_P^E(A, 2m-1) - 1 = \nu_{P,1}(\gamma_A^{2m-1,P}) - 1. \end{aligned} \quad (3.3)$$

Now we compute $i(u^{2m-1})$ via the index iteration method in [Lon1] and [DoL1]. First we recall briefly an index theory for symplectic paths. All the details can be found in [Lon1], [DoL1] and [Liu2].

In the following of this section, we assume P is some matrix of pattern $(-I_{2s-2t}) \diamond I_{2t}$, where $0 \leq t \leq s$.

As usual, the symplectic group $Sp(2n)$ is defined by

$$Sp(2n) = \{M \in GL(2n, \mathbf{R}) \mid M^T J M = J\},$$

whose topology is induced from that of \mathbf{R}^{4n^2} . For $\tau > 0$ we are interested in paths in $Sp(2n)$:

$$\mathcal{P}_\tau(2n) = \{\gamma \in C([0, \tau], Sp(2n)) \mid \gamma(0) = I_{2n}\},$$

which is equipped with the topology induced from that of $Sp(2n)$. The following real function was introduced in [DoL1]:

$$D_{P,\omega}(M) = (-1)^{n-1} \bar{\omega}^n \det(M - \omega P), \quad \forall \omega \in \mathbf{U}, M \in Sp(2n).$$

where \mathbf{U} is the unit circle in the complex plane. Thus for any $\omega \in \mathbf{U}$ the following codimension 1 hypersurface in $Sp(2n)$ is defined in [DoL1]:

$$Sp(2n)_{P,\omega}^0 = \{M \in Sp(2n) \mid D_{P,\omega}(M) = 0\}.$$

For any $M \in Sp(2n)_{P,\omega}^0$, we define a co-orientation of $Sp(2n)_{P,\omega}^0$ at M by the positive direction $\frac{d}{dt} M e^{t\epsilon J} \big|_{t=0}$ of the path $M e^{t\epsilon J}$ with $0 \leq t \leq 1$ and $\epsilon > 0$ being sufficiently small. Let

$$Sp(2n)_{P,\omega}^* = Sp(2n) \setminus Sp(2n)_{P,\omega}^0,$$

$$\mathcal{P}_{P,\tau,\omega}^*(2n) = \{\gamma \in \mathcal{P}_\tau(2n) \mid \gamma_\tau \in Sp(2n)_{P,\omega}^*\},$$

$$\mathcal{P}_{P,\tau,\omega}^0(2n) = \mathcal{P}_\tau(2n) \setminus \mathcal{P}_{P,\tau,\omega}^*(2n).$$

For any two continuous arcs ξ and $\eta : [0, \tau] \rightarrow Sp(2n)$ with $\xi(\tau) = \eta(0)$, it is defined as usual:

$$\eta * \xi(t) = \begin{cases} \xi(2t), & \text{if } 0 \leq t \leq \tau/2, \\ \eta(2t - \tau), & \text{if } \tau/2 \leq t \leq \tau. \end{cases}$$

Given any two $2m_k \times 2m_k$ matrices of square block form $M_k = \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix}$ with $k = 1, 2$, as in [Lon1], the \diamond -product of M_1 and M_2 is defined by the following $2(m_1 + m_2) \times 2(m_1 + m_2)$ matrix $M_1 \diamond M_2$:

$$M_1 \diamond M_2 = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix}$$

Denote by $M^{\diamond k}$ the k -fold \diamond -product $M \diamond \cdots \diamond M$. Note that the \diamond -product of any two symplectic matrices is symplectic. For any two paths $\gamma_j \in \mathcal{P}_\tau(2n_j)$ with $j = 0$ and 1 , let $\gamma_1 \diamond \gamma_2(t) = \gamma_1(t) \diamond \gamma_2(t)$ for all $t \in [0, \tau]$.

A special path $\xi_n \in \mathcal{P}_\tau(2n)$ is defined by

$$\xi_n(t) = \left(\begin{array}{cc} 2 - \frac{t}{\tau} & 0 \\ 0 & (2 - \frac{t}{\tau})^{-1} \end{array} \right)^{\diamond n}, \text{ for } 0 \leq t \leq \tau.$$

Definition 3.3.(cf. [DoL1], also Definition 3.3 of [Liu2]) For any $\omega \in \mathbf{U}$ and $M \in Sp(2n)$, via Definition 5.4.4 in [Lon1], we define

$$\nu_{P,\omega}(M) = \dim_{\mathbf{C}} \ker_{\mathbf{C}}(M - \omega P). \quad (3.4)$$

For any $\tau > 0$ and $\gamma \in \mathcal{P}_\tau(2n)$, define

$$\nu_{P,\omega}(\gamma) = \nu_\omega(\gamma P) = \nu_{P,\omega}(\gamma(\tau)). \quad (3.5)$$

If $\gamma \in \mathcal{P}_{P,\tau,\omega}^*(2n)$, define

$$i_{P,\omega}(\gamma) = [Sp(2n)_{P,\omega}^0 : \gamma * \xi_n], \quad (3.6)$$

where the right hand side of (3.6) is the usual homotopy intersection number, and the orientation of $\gamma * \xi_n$ is its positive time direction under homotopy with fixed end points.

If $\gamma \in \mathcal{P}_{P,\tau,\omega}^0(2n)$, we let $\mathcal{F}(\gamma)$ be the set of all open neighborhoods of γ in $\mathcal{P}_\tau(2n)$, and define

$$i_{P,\omega}(\gamma) = \sup_{U \in \mathcal{F}(\gamma)} \inf \{i_{P,\omega}(\beta) \mid \beta \in U \cap \mathcal{P}_{P,\tau,\omega}^*(2n)\}. \quad (3.7)$$

Then

$$(i_{P,\omega}(\gamma), \nu_{P,\omega}(\gamma)) \in \mathbf{Z} \times \{0, 1, \dots, 2n\}, \quad (3.8)$$

is called the P-index function of γ at ω .

For any $M \in Sp(2n)$ and $\omega \in \mathbf{U}$, the *splitting numbers* $S_M^\pm(P, \omega)$ of M at (P, ω) are defined by

$$S_M^\pm(P, \omega) = \lim_{\epsilon \rightarrow 0^+} i_{P,\omega \exp(\pm \sqrt{-1}\epsilon)}(\gamma) - i_{P,\omega}(\gamma), \quad (3.9)$$

for any path $\gamma \in \mathcal{P}_\tau(2n)$ satisfying $\gamma(\tau) = M$.

Let $\Omega^0(M)$ be the path connected component containing $M = \gamma(\tau)$ of the set

$$\Omega(M) = \{N \in Sp(2n) \mid \sigma(N) \cap \mathbf{U} = \sigma(M) \cap \mathbf{U} \text{ and} \quad (3.10)$$

$$\nu_\lambda(N) = \nu_\lambda(M), \forall \lambda \in \sigma(M) \cap \mathbf{U}\} \quad (3.11)$$

Here $\Omega^0(M)$ is called the *homotopy component* of M in $Sp(2n)$.

In [Lon1], the following symplectic matrices were introduced as basic normal forms:

$$D(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \lambda = \pm 2, \quad (3.12)$$

$$N_1(\lambda, b) = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}, \lambda = \pm 1, b = \pm 1, 0, \quad (3.13)$$

$$R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \theta \in (0, \pi) \cup (\pi, 2\pi), \quad (3.14)$$

$$N_2(\omega, B) = \begin{pmatrix} R(\theta) & B \\ 0 & R(\theta) \end{pmatrix}, \theta \in (0, \pi) \cup (\pi, 2\pi), \quad (3.15)$$

where $B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$ with $b_i \in \mathbf{R}$ and $b_2 \neq b_3$.

Splitting numbers possess the following properties:

Lemma 3.4.(cf. Proposition 3.8 of [DoL1]) *Let $(p_\omega(MP), q_\omega(MP))$ denote the Krein type of MP at ω . For any $M \in Sp(2n)$ and $\omega \in \mathbf{U}$, the splitting numbers $S_M^\pm(P, \omega)$ are well defined and satisfy the following properties.*

(i) $S_M^\pm(P, \omega) = S_{MP}^\pm(\omega)$, where the right-hand side is the splitting numbers given by Definition 9.1.4 of [Lon1].

(ii) $S_M^+(P, \omega) - S_M^-(P, \omega) = p_\omega(MP) - q_\omega(MP)$.

(iii) $S_M^\pm(P, \omega) = S_N^\pm(P, \omega)$ if $NP \in \Omega^0(MP)$.

(iv) $S_{M_1 \diamond M_2}^\pm(P, \omega) = S_{M_1}^\pm(P_1, \omega) + S_{M_2}^\pm(P_2, \omega)$ for $M_j, P_j \in Sp(2n_j)$ with $n_j \in \{1, \dots, n\}$ satisfying $P = P_1 \diamond P_2$ and $n_1 + n_2 = n$.

(v) $S_M^\pm(P, \omega) = 0$ if $\omega \notin \sigma(MP)$.

We have the following

Lemma 3.5. (cf. Theorem 1.8.10 of [Lon1]) *For any $M \in Sp(2n)$, there is a path $f : [0, 1] \rightarrow \Omega^0(M)$ such that $f(0) = M$ and*

$$f(1) = M_1 \diamond \dots \diamond M_l, \quad (3.16)$$

where each M_i is a basic normal form listed in (3.12)-(3.15) for $1 \leq i \leq l$. ■

By Proposition 3.10 of [DoL1], we have the Bott-type formula for (P, ω) -index:

Lemma 3.6. *For any $\gamma \in \mathcal{P}_\tau(2n)$, $z \in \mathbf{U}$ and $m \in \mathbf{N}$, we have*

$$\begin{aligned} i_{P^m, z}(\gamma^{m, P}) &= \sum_{\omega^m = z} i_{P, \omega}(\gamma), \\ \nu_{P^m, z}(\gamma^{m, P}) &= \sum_{\omega^m = z} \nu_{P, \omega}(\gamma). \end{aligned}$$

Now we deduce the index iteration formula for each case in (3.12)-(3.15). Note that the splitting numbers are computed in List 9.1.12 of [Lon1]. Let $M = \gamma(\tau)$.

Case 1. MP is conjugate to a matrix $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ for some $b > 0$.

In this case, we have $(S_M^+(P, 1), S_M^-(P, 1)) = (1, 1)$ by List 9.1.12 of [Lon1] and Lemma 3.4 (i), (iii). Thus by Lemma 3.4 (v) and Lemma 3.6, we have

$$\begin{aligned} i_{P, 1}(\gamma^{2m-1, P}) &= \sum_{\omega^{2m-1} = 1} i_{P, \omega}(\gamma) = \sum_{k=0}^{2m-2} i_{P, e^{2k\pi i/(2m-1)}}(\gamma) = (2m-1)(i_{P, 1}(\gamma) + 1) - 1, \\ \nu_{P, 1}(\gamma^{2m-1, P}) &= 1. \end{aligned} \quad (3.17)$$

Case 2. $MP = I_2$, the 2×2 identity matrix.

In this case, we have $(S_M^+(P, 1), S_M^-(P, 1)) = (1, 1)$ by List 9.1.12 of [Lon1] and Lemma 3.4 (i), (iii). Thus as in Case 1, we have

$$i_{P, 1}(\gamma^{2m-1, P}) = (2m-1)(i_{P, 1}(\gamma) + 1) - 1, \quad \nu_{P, 1}(\gamma^{2m-1, P}) = 2. \quad (3.18)$$

Case 3. MP is conjugate to a matrix $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ for some $b < 0$.

In this case, we have $(S_M^+(P, 1), S_M^-(P, 1)) = (0, 0)$ by List 9.1.12 of [Lon1] and Lemma 3.4 (i), (iii). Thus by Lemma 3.4 (v) and Lemma 3.6, we have

$$\begin{aligned} i_{P,1}(\gamma^{2m-1,P}) &= \sum_{\omega^{2m-1}=1} i_{P,\omega}(\gamma) = \sum_{k=0}^{2m-2} i_{P,e^{2k\pi i/(2m-1)}}(\gamma) = (2m-1)i_{P,1}(\gamma), \\ \nu_{P,1}(\gamma^{2m-1,P}) &= 1. \end{aligned} \quad (3.19)$$

Case 4. MP is conjugate to a matrix $\begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix}$ for some $b < 0$.

In this case, we have $(S_M^+(P, -1), S_M^-(P, -1)) = (1, 1)$ by List 9.1.12 of [Lon1] and Lemma 3.4 (i), (iii). Thus by Lemma 3.4 (v) and Lemma 3.6, we have

$$\begin{aligned} i_{P,1}(\gamma^{2m-1,P}) &= \sum_{\omega^{2m-1}=1} i_{P,\omega}(\gamma) = \sum_{k=0}^{2m-2} i_{P,e^{2k\pi i/(2m-1)}}(\gamma) = (2m-1)i_{P,1}(\gamma), \\ \nu_{P,1}(\gamma^{2m-1,P}) &= 0. \end{aligned} \quad (3.20)$$

Case 5. $MP = -I_2$.

In this case, we have $(S_M^+(P, -1), S_M^-(P, -1)) = (1, 1)$ by List 9.1.12 of [Lon1] and Lemma 3.4 (i), (iii). Thus as in Case 4, we have

$$i_{P,1}(\gamma^{2m-1,P}) = (2m-1)i_{P,1}(\gamma), \quad \nu_{P,1}(\gamma^{2m-1,P}) = 0. \quad (3.21)$$

Case 6. MP is conjugate to a matrix $\begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix}$ for some $b > 0$.

In this case, we have $(S_M^+(P, -1), S_M^-(P, -1)) = (0, 0)$ by List 9.1.12 of [Lon1] and Lemma 3.4 (i), (iii). Thus by Lemma 3.4 (v) and Lemma 3.6, we have

$$i_{P,1}(\gamma^{2m-1,P}) = \sum_{\omega^{2m-1}=1} i_{P,\omega}(\gamma) = \sum_{k=0}^{2m-2} i_{P,e^{2k\pi i/(2m-1)}}(\gamma) = (2m-1)i_{P,1}(\gamma), \quad (3.22)$$

$$\nu_{P,1}(\gamma^{2m-1,P}) = 0. \quad (3.23)$$

Case 7. $MP = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ with some $\theta \in (0, \pi) \cup (\pi, 2\pi)$.

In this case, we have $(S_M^+(P, e^{\sqrt{-1}\theta}), S_M^-(P, e^{\sqrt{-1}\theta})) = (0, 1)$ by List 9.1.12 of [Lon1] and Lemma 3.4 (i), (iii). Thus by Lemma 3.4 (v) and Lemma 3.6, we have

$$i_{P,1}(\gamma^{2m-1,P}) = \sum_{\omega^{2m-1}=1} i_{P,\omega}(\gamma) = \sum_{k=1}^{2m-1} i_{P,e^{2k\pi i/(2m-1)}}(\gamma)$$

$$\begin{aligned}
&= \sum_{0 < 2k < \frac{(2m-1)\theta}{\pi}} i_{P,1}(\gamma) + \sum_{\frac{(2m-1)\theta}{\pi} \leq 2k \leq \frac{(2m-1)(2\pi-\theta)}{\pi}} (i_{P,1}(\gamma) - 1) \\
&+ \sum_{\frac{(2m-1)(2\pi-\theta)}{\pi} < 2k \leq 4m-2} i_{P,1}(\gamma) \\
&= (2m-1)(i_{P,1}(\gamma) - 1) + 2E\left(\frac{(2m-1)\theta}{2\pi}\right) - 1, \\
\nu_{P,1}(\gamma^{2m-1,P}) &= 2 - 2\phi\left(\frac{(2m-1)\theta}{2\pi}\right), \tag{3.24}
\end{aligned}$$

where the function $E(\cdot)$ is defined as in Section 1.

Provided $\theta \in (0, \pi)$. When $\theta \in (\pi, 2\pi)$, we have

$$\begin{aligned}
i_{P,1}(\gamma^{2m-1,P}) &= \sum_{\omega^{2m-1}=1} i_{P,\omega}(\gamma) = \sum_{k=1}^{2m-1} i_{P,e^{2k\pi i/(2m-1)}}(\gamma) \\
&= \sum_{0 < 2k \leq \frac{(2m-1)(2\pi-\theta)}{\pi}} i_{P,1}(\gamma) + \sum_{\frac{(2m-1)(2\pi-\theta)}{\pi} < 2k < \frac{(2m-1)\theta}{\pi}} (i_{P,1}(\gamma) + 1) \\
&+ \sum_{\frac{(2m-1)\theta}{\pi} \leq 2k \leq 4m-2} i_{P,1}(\gamma) \\
&= (2m-1)(i_{P,1}(\gamma) - 1) + 2E\left(\frac{(2m-1)\theta}{2\pi}\right) - 1, \\
\nu_{P,1}(\gamma^{2m-1,P}) &= 2 - 2\phi\left(\frac{(2m-1)\theta}{2\pi}\right). \tag{3.25}
\end{aligned}$$

Case 8. $MP = \begin{pmatrix} R(\theta) & B \\ 0 & R(\theta) \end{pmatrix}$ with some $\theta \in (0, \pi) \cup (\pi, 2\pi)$ and $B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \in \mathbf{R}^{2 \times 2}$, such that $(b_2 - b_3) \sin \theta < 0$.

In this case, we have $(S_M^+(P, e^{\sqrt{-1}\theta}), S_M^-(P, e^{\sqrt{-1}\theta})) = (1, 1)$ by List 9.1.12 of [Lon1] and Lemma 3.4 (i), (iii). Thus by Lemma 3.4 (v) and Lemma 3.6, we have

$$\begin{aligned}
i_{P,1}(\gamma^{2m-1,P}) &= \sum_{\omega^{2m-1}=1} i_{P,\omega}(\gamma) = \sum_{k=1}^{2m-1} i_{P,e^{2k\pi i/(2m-1)}}(\gamma) \\
&= (2m-1)i_{P,1}(\gamma) + 2\phi\left(\frac{(2m-1)\theta}{2\pi}\right) - 2, \\
\nu_{P,1}(\gamma^{2m-1,P}) &= 2 - 2\phi\left(\frac{(2m-1)\theta}{2\pi}\right). \tag{3.26}
\end{aligned}$$

Here from the first line to the second line of Equation (3.26), we used that if $\frac{(2m-1)\theta}{2\pi} \notin \mathbf{N}$, then $i_{P,e^{2k\pi i/(2m-1)}}(\gamma) = i_{P,1}(\gamma)$ for all $1 \leq k \leq 2m-1$ and if $\frac{(2m-1)\theta}{2\pi} \in \mathbf{N}$, then exactly two of the $i_{P,e^{2k\pi i/(2m-1)}}(\gamma)$'s equal to $i_{P,1}(\gamma) - 1$ and the other ones equal to $i_{P,1}(\gamma)$.

Case 9. $MP = \begin{pmatrix} R(\theta) & B \\ 0 & R(\theta) \end{pmatrix}$ with some $\theta \in (0, \pi) \cup (\pi, 2\pi)$ and $B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \in \mathbf{R}^{2 \times 2}$, such that $(b_2 - b_3) \sin \theta > 0$.

In this case, we have $(S_M^+(P, e^{\sqrt{-1}\theta}), S_M^-(P, e^{\sqrt{-1}\theta})) = (0, 0)$ by List 9.1.12 of [Lon1] and Lemma 3.4 (i), (iii). Thus by Lemma 3.4 (v) and Lemma 3.6, we have

$$\begin{aligned} i_{P,1}(\gamma^{2m-1,P}) &= \sum_{\omega^{2m-1}=1} i_{P,\omega}(\gamma) = \sum_{k=1}^{2m-1} i_{P,e^{2k\pi i/(2m-1)}}(\gamma) = (2m-1)i_{P,1}(\gamma), \\ \nu_{P,1}(\gamma^{2m-1,P}) &= 2 - 2\phi\left(\frac{(2m-1)\theta}{2\pi}\right), \end{aligned} \quad (3.27)$$

Case 10. MP is hyperbolic, i.e., $\mathbf{U} \cap \sigma(MP) = \emptyset$.

In this case, by Lemma 3.4 (v) and Lemma 3.6, we have

$$\begin{aligned} i_{P,1}(\gamma^{2m-1,P}) &= \sum_{\omega^{2m-1}=1} i_{P,\omega}(\gamma) = \sum_{k=1}^{2m-1} i_{P,e^{2k\pi i/(2m-1)}}(\gamma) = (2m-1)i_{P,1}(\gamma), \\ \nu_{P,1}(\gamma^{2m-1,P}) &= 0. \end{aligned} \quad (3.28)$$

The following new theorem will play a crucial role in our proof of Theorem 1.1:

Theorem 3.7. *Suppose u^{2m-1} is a nonzero critical point of Ψ_a such that u corresponds to a P -symmetric closed characteristic (τ, y) . Then we have $i(u^3) - 3i(u) \leq 2\kappa + 2n$ and $i(u^3) + \nu(u^3) - 3(i(u) + \nu(u)) \geq 2\kappa + 2 - 2n$. In particular, we have the following*

- (i) if $i(u^3) - 3i(u) \geq 2n$, then $e(\tau, y) \geq 2n - 2\kappa$.
- (ii) if $i(u^3) + \nu(u^3) - 3(i(u) + \nu(u)) \leq 6\kappa + 2 - 2n$, then $e(\tau, y) \geq 2n - 4\kappa$.

Proof. Firstly, we compute $i_{P,1}(\gamma^{3,P}) - 3i_{P,1}(\gamma)$ and $i_{P,1}(\gamma^{3,P}) + \nu_{P,1}(\gamma^{3,P}) - 3(i_{P,1}(\gamma) + \nu_{P,1}(\gamma))$ for any symplectic path $\gamma \in \mathcal{P}_\tau(2n)$ satisfying $\gamma(\tau) = M$. We consider each of the above cases.

Case 1. MP is conjugate to a matrix $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ for some $b > 0$.

In this case, we have

$$\begin{aligned} i_{P,1}(\gamma^{3,P}) - 3i_{P,1}(\gamma) &= 3(i_{P,1}(\gamma) + 1) - 1 - 3i_{P,1}(\gamma) = 2, \\ i_{P,1}(\gamma^{3,P}) + \nu_{P,1}(\gamma^{3,P}) - 3(i_{P,1}(\gamma) + \nu_{P,1}(\gamma)) &= 3(i_{P,1}(\gamma) + 1) - 3(i_{P,1}(\gamma) + 1) = 0. \end{aligned}$$

Case 2. $MP = I_2$, the 2×2 identity matrix.

In this case, we have

$$\begin{aligned} i_{P,1}(\gamma^{3,P}) - 3i_{P,1}(\gamma) &= 3(i_{P,1}(\gamma) + 1) - 1 - 3i_{P,1}(\gamma) = 2, \\ i_{P,1}(\gamma^{3,P}) + \nu_{P,1}(\gamma^{3,P}) - 3(i_{P,1}(\gamma) + \nu_{P,1}(\gamma)) &= 3(i_{P,1}(\gamma) + 1) + 1 - 3(i_{P,1}(\gamma) + 2) = -2. \end{aligned}$$

Case 3. MP is conjugate to a matrix $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ for some $b < 0$.

In this case, we have

$$\begin{aligned} i_{P,1}(\gamma^{3,P}) - 3i_{P,1}(\gamma) &= 3i_{P,1}(\gamma) - 3i_{P,1}(\gamma) = 0, \\ i_{P,1}(\gamma^{3,P}) + \nu_{P,1}(\gamma^{3,P}) - 3(i_{P,1}(\gamma) + \nu_{P,1}(\gamma)) &= 3i_{P,1}(\gamma) + 1 - 3(i_{P,1}(\gamma) + 1) = -2. \end{aligned}$$

Case 4. MP is conjugate to a matrix $\begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix}$ for some $b < 0$.

In this case, we have

$$\begin{aligned} i_{P,1}(\gamma^{3,P}) - 3i_{P,1}(\gamma) &= 3i_{P,1}(\gamma) - 3i_{P,1}(\gamma) = 0, \\ i_{P,1}(\gamma^{3,P}) + \nu_{P,1}(\gamma^{3,P}) - 3(i_{P,1}(\gamma) + \nu_{P,1}(\gamma)) &= 3i_{P,1}(\gamma) - 3i_{P,1}(\gamma) = 0. \end{aligned}$$

Case 5. $MP = -I_2$.

In this case, we have

$$\begin{aligned} i_{P,1}(\gamma^{3,P}) - 3i_{P,1}(\gamma) &= 3i_{P,1}(\gamma) - 3i_{P,1}(\gamma) = 0, \\ i_{P,1}(\gamma^{3,P}) + \nu_{P,1}(\gamma^{3,P}) - 3(i_{P,1}(\gamma) + \nu_{P,1}(\gamma)) &= 3i_{P,1}(\gamma) - 3i_{P,1}(\gamma) = 0. \end{aligned}$$

Case 6. MP is conjugate to a matrix $\begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix}$ for some $b > 0$.

In this case, we have

$$\begin{aligned} i_{P,1}(\gamma^{3,P}) - 3i_{P,1}(\gamma) &= 3i_{P,1}(\gamma) - 3i_{P,1}(\gamma) = 0, \\ i_{P,1}(\gamma^{3,P}) + \nu_{P,1}(\gamma^{3,P}) - 3(i_{P,1}(\gamma) + \nu_{P,1}(\gamma)) &= 3i_{P,1}(\gamma) - 3i_{P,1}(\gamma) = 0. \end{aligned}$$

Case 7. $MP = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ with some $\theta \in (0, \pi) \cup (\pi, 2\pi)$.

In this case, we have

$$\begin{aligned} i_{P,1}(\gamma^{3,P}) - 3i_{P,1}(\gamma) &= 3(i_{P,1}(\gamma) - 1) + 2E\left(\frac{3\theta}{2\pi}\right) - 1 - 3i_{P,1}(\gamma) \leq 2, \\ i_{P,1}(\gamma^{3,P}) + \nu_{P,1}(\gamma^{3,P}) - 3(i_{P,1}(\gamma) + \nu_{P,1}(\gamma)) &= \left(3(i_{P,1}(\gamma) - 1) + 2E\left(\frac{3\theta}{2\pi}\right) - 1 + 2 - 2\phi\left(\frac{3\theta}{2\pi}\right)\right) - 3i_{P,1}(\gamma) \\ &= -2 + 2E\left(\frac{3\theta}{2\pi}\right) - 2\phi\left(\frac{3\theta}{2\pi}\right) \geq -2. \end{aligned}$$

Case 8. $MP = \begin{pmatrix} R(\theta) & B \\ 0 & R(\theta) \end{pmatrix}$ with some $\theta \in (0, \pi) \cup (\pi, 2\pi)$ and $B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \in \mathbf{R}^{2 \times 2}$,

such that $(b_2 - b_3) \sin \theta < 0$.

In this case, we have

$$\begin{aligned} i_{P,1}(\gamma^{3,P}) - 3i_{P,1}(\gamma) &= 3i_{P,1}(\gamma) + 2\phi\left(\frac{3\theta}{2\pi}\right) - 2 - 3i_{P,1}(\gamma) \leq 0, \\ i_{P,1}(\gamma^{3,P}) + \nu_{P,1}(\gamma^{3,P}) - 3(i_{P,1}(\gamma) + \nu_{P,1}(\gamma)) &= 3i_{P,1}(\gamma) - 3i_{P,1}(\gamma) = 0. \end{aligned}$$

Case 9. $MP = \begin{pmatrix} R(\theta) & B \\ 0 & R(\theta) \end{pmatrix}$ with some $\theta \in (0, \pi) \cup (\pi, 2\pi)$ and $B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \in \mathbf{R}^{2 \times 2}$, such that $(b_2 - b_3) \sin \theta > 0$.

In this case, we have

$$\begin{aligned} i_{P,1}(\gamma^{3,P}) - 3i_{P,1}(\gamma) &= 3i_{P,1}(\gamma) - 3i_{P,1}(\gamma) = 0, \\ i_{P,1}(\gamma^{3,P}) + \nu_{P,1}(\gamma^{3,P}) - 3(i_{P,1}(\gamma) + \nu_{P,1}(\gamma)) &= 3i_{P,1}(\gamma) + 2 - 2\phi\left(\frac{3\theta}{2\pi}\right) - 3i_{P,1}(\gamma) \geq 0. \end{aligned}$$

Case 10. MP is hyperbolic, i.e., $\mathbf{U} \cap \sigma(MP) = \emptyset$.

In this case, we have

$$\begin{aligned} i_{P,1}(\gamma^{3,P}) - 3i_{P,1}(\gamma) &= 3i_{P,1}(\gamma) - 3i_{P,1}(\gamma) = 0, \\ i_{P,1}(\gamma^{3,P}) + \nu_{P,1}(\gamma^{3,P}) - 3(i_{P,1}(\gamma) + \nu_{P,1}(\gamma)) &= 3i_{P,1}(\gamma) - 3i_{P,1}(\gamma) = 0. \end{aligned}$$

Now for a P-symmetric closed characteristic (τ, y) , H_2, τ_2 is defined as in Lemma 3.1. Its fundamental solution $\gamma = \gamma_y : [0, \tau_2/2] \rightarrow Sp(2n)$ with $\gamma_y(0) = I_{2n}$ of the linearized Hamiltonian system

$$\dot{\gamma}_y(t) = JH_2''(y(t))\gamma_y(t), \quad \forall t \in \mathbf{R}, \quad (3.29)$$

is called the *associated symplectic path* of P-symmetric closed characteristic (τ, y) . By Lemma 3.5, we suppose $\gamma(\frac{\tau_2}{2})P$ is conjugate to $N_1(1, 1)^{\diamond p_-} \diamond N_1(1, -1)^{\diamond p_+} \diamond (I_{2p_0}) \diamond R(\theta_1) \diamond \cdots \diamond R(\theta_r) \diamond N_2(\omega_1, B_1) \diamond \cdots \diamond N_2(\omega_s, B_s) \diamond M_0$ with $\sigma(M_0) \cap (\mathbf{U} - \{-1\}) = \emptyset$. Combining the above cases 1-10, we obtain

$$\begin{aligned} i_{P,1}(\gamma^{3,P}) - 3i_{P,1}(\gamma) &\leq 2p_- + 2p_0 + 2r \leq 2n, \\ i_{P,1}(\gamma^{3,P}) + \nu_{P,1}(\gamma^{3,P}) - 3(i_{P,1}(\gamma) + \nu_{P,1}(\gamma)) &\geq -2p_0 - 2p_+ - 2r \geq -2n. \end{aligned} \quad (3.30)$$

And by Theorem 3.2, we have $\nu(u^3) = \nu_{P,1}(\gamma^{3,P}) - 1$, $\nu(u) = \nu_{P,1}(\gamma) - 1$, $i(u^3) = i_{P,1}(\gamma^{3,P}) - \kappa$, $i(u) = i_{P,1}(\gamma) - \kappa$. Then we obtain

$$\begin{aligned} i(u^3) - 3i(u) &= 2\kappa + i_{P,1}(\gamma^{3,P}) - 3i_{P,1}(\gamma) \\ &\leq 2\kappa + 2p_- + 2p_0 + 2r \leq 2\kappa + 2n, \end{aligned} \quad (3.31)$$

and

$$\begin{aligned}
& i(u^3) + \nu(u^3) - 3(i(u) + \nu(u)) \\
& = 2\kappa + 2 + i_{P,1}(\gamma^{3,P}) + \nu_{P,1}(\gamma^{3,P}) - 3(i_{P,1}(\gamma) + \nu_{P,1}(\gamma)) \\
& \geq 2\kappa + 2 - 2p_0 - 2p_+ - 2r \geq 2\kappa + 2 - 2n.
\end{aligned} \tag{3.32}$$

If $i(u^3) - 3i(u) \geq 2n$, then by (3.31), we have

$$2p_- + 2p_0 + 2r \geq 2n - 2\kappa, \tag{3.33}$$

i.e. $e(\gamma(\frac{\tau_2}{2})P) \geq 2n - 2\kappa$.

If $i(u^3) + \nu(u^3) - 3(i(u) + \nu(u)) \leq 6\kappa + 2 - 2n$, then by (3.32), we have

$$2p_0 + 2p_+ + 2r \geq 2n - 4\kappa, \tag{3.34}$$

i.e. $e(\gamma(\frac{\tau_2}{2})P) \geq 2n - 4\kappa$. Noticing that $e(\tau, y) = e(\gamma(\tau_2)) = e((\gamma(\frac{\tau_2}{2})P)^2)$, we complete our proof.

4 Proof of the main theorem

In this section, we give the proof of the main theorem.

Firstly, we point out a minor error in Example 6 on Page 278 of [DoL2] which is useful for us:

Lemma 4.1. *For any $c > 0$, we have*

$$i_P^E(cI_{2n}|_{[0,s]}) = 2\kappa \left(E\left(\frac{cs}{2\pi}\right) - 1 \right) + 2(n - \kappa) \left(E\left(\frac{cs + \pi}{2\pi}\right) - 1 \right). \tag{4.1}$$

Proof. Note that the definition of $E(a)$ in our paper is different from that in [DoL2] by 1. In the proof of Example 6 of [DoL2], $t = s_k = (2k\pi + \frac{3}{2}\pi)/c$ should be changed into $t = s_k = (2k\pi + \pi)/c$, and the expression in (3.14) of [DoL2] is wrong. One can easily verify our expression in (4.1) is right.

Note that since $H_2(\cdot)$ is positive homogeneous of degree-two, by the (r, R) -pinched condition we have

$$|x|^2 R^{-2} \leq H_2(x) \leq |x|^2 r^{-2}, \quad \forall x \in \Sigma \tag{4.2}$$

Comparing with the theorem of Morse-Schoenberg in the study of geodesics, we have the following

Proposition 4.2. *Let $\Sigma \in \mathcal{H}_\kappa(2n)$ which is (r, R) -pinched. Suppose u^{2m-1} is a nonzero critical point of Ψ_a such that u corresponds to a P -symmetric closed characteristic (τ, y) , and $\tau_2 = A(\tau, y)$ as in Lemma 3.1. Then we have the following*

$$i(u^{2m-1}) \geq 2nl, \quad \text{if } \frac{(2m-1)\tau_2}{2} > l\pi R^2; \quad (4.3)$$

$$i(u^{2m-1}) + \nu(u^{2m-1}) \leq 2n(l-1) - 1, \quad \text{if } \frac{(2m-1)\tau_2}{2} < (l - \frac{1}{2})\pi r^2, \quad (4.4)$$

for some $l \in \mathbf{N}$.

Proof. Consider the following three quadratic forms on $L_\kappa^2(0, (2m-1)\tau_2/2)$

$$\begin{aligned} q_{(2m-1)\tau_2/2, \kappa}^R(v, v) &:= \int_0^{(2m-1)\tau_2/2} [(Jv, \Pi_{(2m-1)\tau_2/2, \kappa} v) + (\frac{R^2}{2} Jv, Jv)] dt \\ q_{(2m-1)\tau_2/2, \kappa}(v, v) &:= \int_0^{(2m-1)\tau_2/2} [(Jv, \Pi_{(2m-1)\tau_2/2, \kappa} v) + (H_2''(y(t))^{-1} Jv, Jv)] dt \\ q_{(2m-1)\tau_2/2, \kappa}^r(v, v) &:= \int_0^{(2m-1)\tau_2/2} [(Jv, \Pi_{(2m-1)\tau_2/2, \kappa} v) + (\frac{r^2}{2} Jv, Jv)] dt \end{aligned}$$

By the (r, R) -pinched condition, we have

$$q_{(2m-1)\tau_2/2, \kappa}^R(v, v) \geq q_{(2m-1)\tau_2/2, \kappa}(v, v) \geq q_{(2m-1)\tau_2/2, \kappa}^r(v, v).$$

Thus we have $i_{(2m-1)\tau_2/2}^R \leq i_{(2m-1)\tau_2/2} \leq i_{(2m-1)\tau_2/2}^r$, where $i_{(2m-1)\tau_2/2}^R$, $i_{(2m-1)\tau_2/2}$ and $i_{(2m-1)\tau_2/2}^r$ denote the indices of $q_{(2m-1)\tau_2/2, \kappa}^R$, $q_{(2m-1)\tau_2/2, \kappa}$ and $q_{(2m-1)\tau_2/2, \kappa}^r$ respectively. By Lemma 3.1, Lemma 4.1 and the condition in (4.3), we obtain

$$\begin{aligned} i_{(2m-1)\tau_2/2}^R &= i_P^E(\frac{2}{R^2} I_{2n}, 2m-1) \\ &= 2\kappa \left(E(\frac{\frac{2}{R^2}(2m-1)\tau_2/2}{2\pi}) - 1 \right) + 2(n-\kappa) \left(E(\frac{\frac{2}{R^2}(2m-1)\tau_2/2 + \pi}{2\pi}) - 1 \right) \\ &\geq 2nl, \end{aligned} \quad (4.5)$$

$$i(u^{2m-1}) = i_{(2m-1)\tau_2/2} \geq i_{(2m-1)\tau_2/2}^R \geq 2nl. \quad (4.6)$$

Hence (4.3) holds. Denote by the orthogonal splitting $L_\kappa^2(0, (2m-1)\tau_2/2) = E_- \oplus E_0 \oplus E_+$ of $L_\kappa^2(0, (2m-1)\tau_2/2)$ into negative, zero and positive subspaces. Then we have the following observation: If V is a subspace of $L_\kappa^2(0, (2m-1)\tau_2/2)$ such that $q_{(2m-1)\tau_2/2, \kappa}$ is negative semi-definite, i.e., $v \in V$ implies $q_{(2m-1)\tau_2/2, \kappa}(v, v) \leq 0$, then $\dim V \leq \dim E_- + \dim E_0$. In fact, this is a simple fact of linear algebra: Let

$$pr_- : L_\kappa^2(0, (2m-1)\tau_2/2) = E_- \oplus E_0 \oplus E_+ \rightarrow E_-$$

be the orthogonal projection. Consider $pr_-|V : V \rightarrow E_-$. Then $v \in \ker(pr_-|V)$ must belong to E_0 . That is, since $q_{(2m-1)\tau_2/2, \kappa}(v, v) > 0, 0 \neq v \in E_+$. From

$$\dim V = \dim \operatorname{Im}(pr_-|V) + \dim \ker(pr_-|V)$$

we prove our claim.

Let $\epsilon > 0$ be small enough such that $\frac{(2m-1)\tau_2}{2} < (l - \frac{1}{2})\pi(r - \epsilon)^2$. If V is a subspace of $L_\kappa^2(0, (2m-1)\tau_2/2)$ such that $q_{(2m-1)\tau_2/2, \kappa}|V \leq 0$, then $q_{(2m-1)\tau_2/2, \kappa}^{r-\epsilon}(v, v) < 0, 0 \neq v \in V$. Thus we have $\dim V \leq i_{(2m-1)\tau_2/2}^{r-\epsilon}$. In particular, by Lemma 3.1, we have

$$i(u^{2m-1}) + \nu(u^{2m-1}) = i_{(2m-1)\tau_2/2} + \nu(q_{(2m-1)\tau_2/2, \kappa}) - 1 \leq i_{(2m-1)\tau_2/2}^{r-\epsilon} - 1. \quad (4.7)$$

On the other hand, similarly to (4.5), by the condition in (4.4), we have

$$\begin{aligned} i_{(2m-1)\tau_2/2}^{r-\epsilon} &= 2\kappa \left(E\left(\frac{\frac{2}{(r-\epsilon)^2}(2m-1)\tau_2/2}{2\pi}\right) - 1 \right) + 2(n - \kappa) \left(E\left(\frac{\frac{2}{(r-\epsilon)^2}(2m-1)\tau_2/2 + \pi}{2\pi}\right) - 1 \right) \\ &\leq 2n(l - 1). \end{aligned} \quad (4.8)$$

Hence, $i(u^{2m-1}) + \nu(u^{2m-1}) \leq 2n(l - 1) - 1$. The proof is complete.

By Theorem 1.1 of [LiZ1], we have:

Lemma 4.3. *Assume $\Sigma \in \mathcal{H}_\kappa(2n)$ and $0 < r \leq |x| \leq R$, $\forall x \in \Sigma$ with $\frac{R}{r} < \sqrt{2}$. Then there exist at least $n - \kappa$ geometrically distinct P -symmetric closed characteristics (τ_i, y_i) on Σ , where τ_i is the minimal period of y_i , and the actions $A(\tau_i, y_i)$ satisfy:*

$$\pi r^2 \leq A(\tau_i, y_i) \leq \pi R^2, \forall 1 \leq i \leq n - \kappa. \quad (4.9)$$

By the proof of the above Lemma and Proposition 2.2, we have

Theorem 4.4. *Let $\{(\tau_1, y_1), \dots, (\tau_{n-\kappa}, y_{n-\kappa})\}$ be the P -symmetric closed characteristics found in the above Lemma. Then we have*

$$\Psi'_a(u_i) = 0, \quad i(u_i) \leq 2(i - 1) \leq i(u_i) + \nu(u_i) - 1. \quad (4.10)$$

for $1 \leq i \leq n - \kappa$, where u_i is the unique critical point of Ψ_a corresponding to (τ_i, y_i) .

Now we give the proof of the main theorem.

Proof of Theorem 1.1. Let $\Sigma \in \mathcal{H}_\kappa(2n)$ which is (r, R) -pinched with $\frac{R}{r} < \sqrt{\frac{5}{3}}$, then by (4.2) we have

$$r \leq |x| \leq R, \forall x \in \Sigma \quad (4.11)$$

Thus by Theorem 4.4, we obtain $n - \kappa$ geometrically distinct prime P-symmetric closed characteristics $\{(\tau_1, y_1), \dots, (\tau_{n-\kappa}, y_{n-\kappa})\}$ such that (4.10) hold.

In the following, we prove $e(\tau_i, y_i) \geq 2n - 4\kappa$ for $i = 1, n - \kappa$.

Note that we always have $e(\tau_1, y_1) \geq 2n - 4\kappa$ by the proof of Theorem 1 of [DoL2], $\sqrt{\frac{5}{3}}$ -pinching is not necessary.

Note that we can prove $e(\tau_1, y_1) \geq 2n - 2\kappa$ under $\sqrt{\frac{3}{2}}$ -pinching condition. In fact, by (4.10), we have $i(u_1) = 0$. On the other hand, from (4.9), we have

$$\frac{(4-1)}{2}A(\tau_1, y_1) \geq \frac{3}{2}\pi r^2 > \pi R^2,$$

where we used the pinching condition $\frac{R}{r} < \sqrt{\frac{3}{2}}$. By Proposition 4.2, we obtain

$$i(u_1^3) \geq 2n,$$

then

$$i(u_1^3) - 3i(u_1) \geq 2n.$$

By Theorem 3.7 (i), we get $e(\tau_1, y_1) \geq 2n - 2\kappa$.

Now we prove $e(\tau_{n-\kappa}, y_{n-\kappa}) \geq 2n - 4\kappa$. In fact, by (4.10), we have

$$2(n - \kappa) - 1 \leq i(u_{n-\kappa}) + \nu(u_{n-\kappa}). \quad (4.12)$$

On the other hand, from (4.9), we have

$$\frac{(4-1)}{2}A(\tau_{n-\kappa}, y_{n-\kappa}) \leq \frac{3}{2}\pi R^2 < \frac{5}{2}\pi r^2 = (3 - \frac{1}{2})\pi r^2.$$

By Proposition 4.2, we obtain

$$i(u_{n-\kappa}^3) + \nu(u_{n-\kappa}^3) \leq 4n - 1, \quad (4.13)$$

Combining (4.12) with (4.13), we obtain

$$i(u_{n-\kappa}^3) + \nu(u_{n-\kappa}^3) - 3(i(u_{n-\kappa}) + \nu(u_{n-\kappa})) \leq 6\kappa + 2 - 2n, \quad (4.14)$$

then $e(\tau_{n-\kappa}, y_{n-\kappa}) \geq 2n - 4\kappa$ follows from Theorem 3.7 (ii). The proof is complete.

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